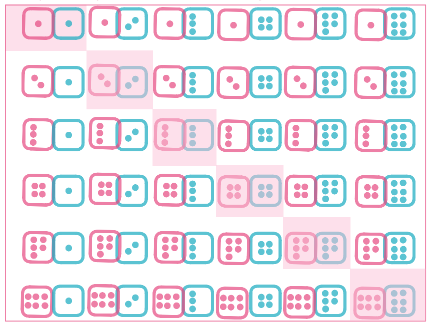
7 - Counting (continued)

Sample space of rolling two dice:



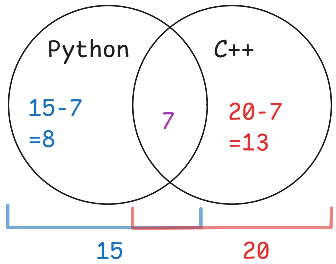
Example 1. Roll two dice. How many outcomes have the 2nd dice roll greater than the 1st dice roll?

Answer 1: To get the upper triangle of the sample space (right), subtracting the diagonal (6 outcomes) and divide by 2.

This gives $\frac{6^2-6}{2} = 15$ outcomes.

Answer 2: Sample two numbers without replacement from {1, 2, ..., 6} ("6 choose 2" = 15 ways), like {3, 1}, {4, 2}. Now note that any unordered set admits a natural order: increasing order, so {3, 1} -> (1 < 3).

Example 2. Of 30 students, 20 dislike Python, 15 dislike C++, and 7 dislike Python and C++. How many dislike Python or C++?



Answer: By PIE (below), the answer is 20+15-7 (since 20+7 counts the intersection of 7 students twice).

Principle of inclusion-exclusion. (PIE) $\vert A \cup B\vert = \vert A\vert + \vert B\vert - \vert A \cap B\vert$

Two events A and B are **mutually exclusive** or **disjoint** if $A \cap B = \varnothing$, in which case $\vert A \cap B\vert = 0$ and $\vert A \cup B\vert = \vert A\vert + \vert B\vert$

Example 3. How many cards have black face or red number in a standard card deck?

Black face cards are ♠J,♠Q,♠K,♣J,♣Q,♣K totaling 6 cards

Red number cards are A♥,2♥,3♥,...,10♥,A♦,2♦,3♦,...,10♦ totaling 20 cards

Red & black cards are mutually exclusive. Overall, there are 20+6 such cards.

Example 4. How many numbers from 1 to 100 are not divisible by 5?

Answer: "Count by complement" since the complement {numbers divisible by 5} is easier to count:

{Multiples of 5's in 1, ..., 100} ∪ {Non-multiples of 5's in 1, ..., 100} = {1, 2, ..., 100}

Size 20 So size 100-20 Size 100

disjoint

8 - Probability

Probability of an event is how often it occurs in the long run:

$$P(\text{Event}) = \lim_{n\to\infty} \frac{\text{Occurrences of the event in } n \text{ repeated experiments}}{n}$$

Law of Large Numbers. If each outcome is equally likely to occur, then

$$P(\text{Event}) = \frac{\#\text{ of outcomes in event}}{\#\text{ of outcomes in sample space}}$$

Example 5. Find the probability each occurs (in the long run):

(a) Get heads when flipping a fair coin.

1/2 since event = {H} and sample space = {H,T}

(b) Roll two fair dice and getting 2nd roll > 1st roll.

15/36 since this event has size 15 out of sample space of size 6x6 = 36.

(c) In Ex 2: pick a student (via SRS) who dislikes Python and C++.

(d) From a standard deck, draw a card that has black face or red number. $\frac{26}{52} = \frac{1}{2}$

(e) In a poker hand of 5 cards, find the probability of holding 2 aces and 3 jacks.

Answer: Sample space = {Having 5 cards from 52 cards} ("52 choose 5" outcomes)
Event = {Have 2 aces and 3 jacks}
= {Have 2 aces from 4} x {Have 3 J from 4} (6 x 4 = 24 outcomes)

Answer: $\frac{24}{\binom{5}{2}} \approx 0.9 \times 10^{-5}$

Exercises. Find the probability of the following events:

1. Getting a total of 7 or 11 when a pair of fair dice is tossed.
2. Getting a number that ends in 1 from 1, 2, 3, ..., 100 by using a random number generator.
3. Getting a number whose English words has the letter "x" from 1, 2, 3, ..., 100 by using a random number generator.
4. Getting (at least) two sixes in a row when rolling 3 fair dice? (JP Morgan interview, easy)
5. Observing at least three heads in 10 flips of a fair coin.

Axioms of Probability. Given sample space S, given events E and F:

Axiom 1: $0 \leq P(E) \leq 1$ All probabilities are numbers between 0 and 1.

Axiom 2: $P(S) = 1$ All outcomes must be from the Sample Space.

Axiom 3: If E and F are mutually exclusive, then

$$P(E \text{ or } F) = P(E \cup F) = P(E) + P(F) \quad \text{and} \quad P(E \text{ and } F) = P(E \cap F) = 0$$

Principle of inclusion-exclusion: (Consequence)

$$P(E \text{ or } F) = P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Probability of complement: (Consequence) If E and E^c are complements, then:

$$P(E^c) = 1 - P(E)$$